**Internationales Mechanik-Kolloquium
zum Abschied von
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Zur Integration zeitabhängiger PDE-Systeme

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PDE-System (3D)

$$\partial_t \mathbf{u} \;=\; \mathbf{A}^x \partial_x \mathbf{u} + \mathbf{A}^y \partial_y \mathbf{u} + \mathbf{A}^z \partial_z \mathbf{u} \quad (+\mathbf{q} + \cdots)$$

Spaltenmatrix u, quatr. Matrizen $\mathbf{A}^i$
partielle Differentiationen $\partial_t \equiv \frac{\partial}{\partial t}$, $\partial_i \equiv \frac{\partial}{\partial x^i}$

$$u_j \;=\; u_j(t, x, y, z)$$
$$\mathbf{u} \;=\; \mathbf{u}(t, \underset{\sim}{\mathbf{r}})$$

$$a^i_j \;=\; a^i_j(t, \underset{\sim}{\mathbf{r}}, \mathbf{u})$$

$$\partial_t \mathbf{u} \;=\; \underset{\sim}{\mathbf{A}} \underset{\sim}{\boldsymbol{\nabla}} \mathbf{u} \quad (+ \cdots)$$

Gewöhnliche Differentialgleichung (DG)

$$\frac{du}{dt} = pu \qquad (\; +q \;)$$

$$u = u(t), p = p(t, u)$$

Lösung:

$$u(t) \; = \; e^{\int_{t_0}^{t} p(\tau) d\tau} \; u(t_0)$$

Partielle DG (PDE)

$$\partial_t u = a\partial_x u$$

$$u = u(t,x), a = a(t,x,u)$$

Lösung PDE: setze hier Funktion $p = a\partial_x$ als Funktionsoperator

$$u(t,x) \; = \; e^{\int_{t_0}^{t} a(\tau)\partial_x d\tau} \; u(t_0,x)$$

Taylor Reihe

$$
\begin{aligned}
u(x+h^x) &= u(x) + h^x \partial_x u(x) + \tfrac{1}{2} h^{x2} \partial_x^2 u(x) + \cdots \\
&= \Big(1 + h^x \partial_x + \tfrac{1}{2} h^{x2} \partial_x^2 + \cdots \Big) u(x)
\end{aligned}
$$

$$u(x+h^x) = e^{h^x \partial_x} u(x)$$

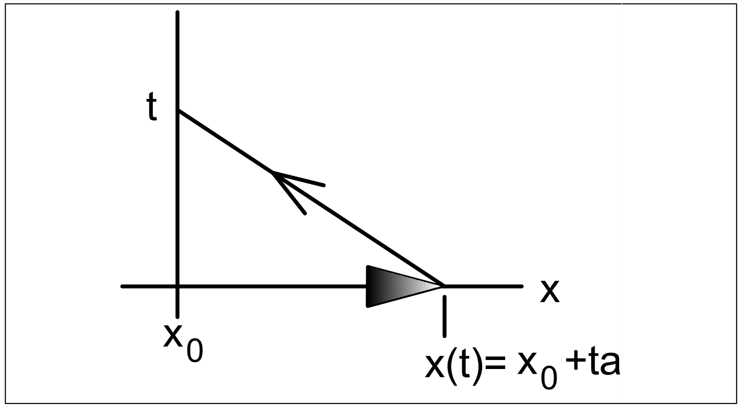
allgemein räumlich:

$$u(\underset{\sim}{\mathbf{r}} + \underset{\sim}{\mathbf{h}}) = e^{\underset{\sim}{\mathbf{h}} \underset{\sim}{\boldsymbol{\nabla}}} u(\underset{\sim}{\mathbf{r}})$$

Lösung PDE

$$h^x = t\bar{a} = \int_{t_0}^{t} a(\tau) d\tau$$

$$\begin{aligned} u(t,x) &= e^{h^x \partial_x} u(t_0, x) \\ &= u(t_0, x + h^x) \end{aligned}$$



Figur 1: 1D Lösung

Partielle DG (PDE), räumlich

$$\partial_t u \;=\; \underset{\sim}{\mathbf{a}}\underset{\sim}{\boldsymbol{\nabla}} u$$

$$u = u(t, \underset{\sim}{\mathbf{r}}), \underset{\sim}{\mathbf{a}} = \underset{\sim}{\mathbf{a}}(t, \underset{\sim}{\mathbf{r}}, u)$$

z.B. 2D karthesisch

$$\partial_t u \;=\; a^x \partial_x u + a^y \partial_y u$$

Lösung PDE, räumlich: setze hier Funktion $p = \underset{\sim}{\mathbf{a}}\underset{\sim}{\boldsymbol{\nabla}}$ als Funktionsoperator

$$u(t, \underset{\sim}{\mathbf{r}}) \;=\; e^{\int_{t_0}^{t} \underset{\sim}{\mathbf{a}}(\tau)\underset{\sim}{\boldsymbol{\nabla}}\, d\tau}\, u(t_0, \underset{\sim}{\mathbf{r}})$$

$$\begin{aligned} u(t, \underset{\sim}{\mathbf{r}}) \;&=\; e^{\underset{\sim}{\mathbf{h}}\underset{\sim}{\boldsymbol{\nabla}}} u(t_0, \underset{\sim}{\mathbf{r}}) \\ &=\; u(t_0, \underset{\sim}{\mathbf{r}} + \underset{\sim}{\mathbf{h}}) \end{aligned}$$

$$\underset{\sim}{\mathbf{h}} = t\underset{\sim}{\bar{\mathbf{a}}} = \int_{t_0}^{t} \underset{\sim}{\mathbf{a}}(\tau) d\tau$$

z.B. 2D karthesisch

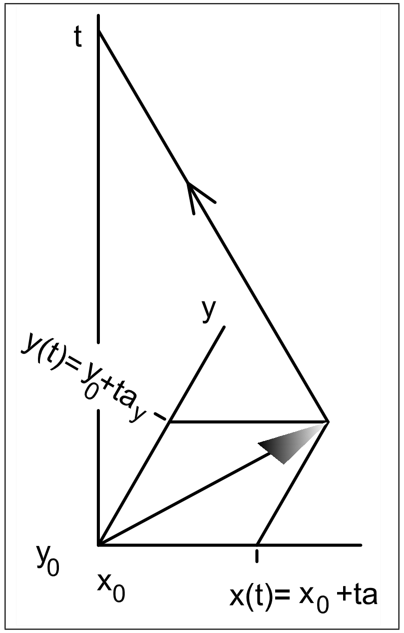
$$
\begin{aligned}
u(t,x,y) &= e^{\int\limits_{t_0}^{t}\left(a^x(\tau)\partial_x + a^y(\tau)\partial_y\right)d\tau} u(t_0,x,y) \\
&= e^{\left(h^x\partial_x + h^y\partial_y\right)} u(t_0,x,y)
\end{aligned}
$$

kommutativ

$$
\begin{aligned}
&= e^{h^x\partial_x}\left(e^{h^y\partial_y} u(t_0,x,y)\right) \\
&= e^{h^x\partial_x} u(t_0,x,y+h^y) \\
&= e^{h^y\partial_y} u(t_0,x+h^x,y)
\end{aligned}
$$

z.B. 2D karthesisch

$$u(t,x,y) \;=\; u(t_0, x + h^x, y + h^y)$$



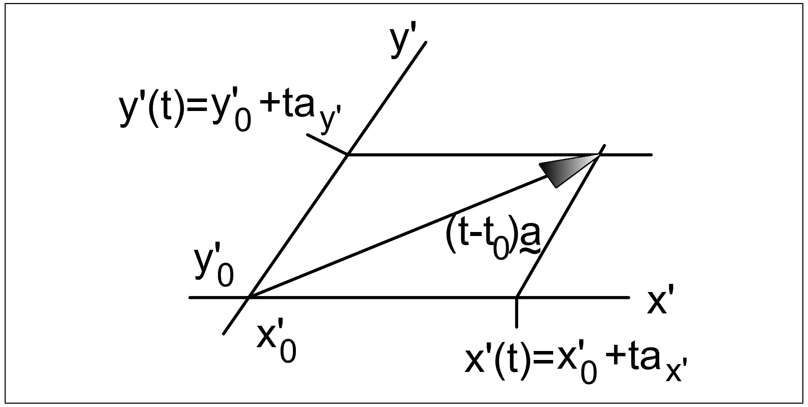
Figur 2: 2D Lösung, karthesisch

z.B. 2D allgemein

$$\partial_t u = \underset{\sim}{\mathbf{a}} \underset{\sim}{\boldsymbol{\nabla}} u$$

$$\underset{\sim}{\mathbf{r}} = \underset{\sim}{\mathbf{r_0}} + (t - t_0)\underset{\sim}{\mathbf{a}}$$

$$u(t, x', y') = u(t_0, x' + h^{x'}, y' + h^{y'})$$



Figur 3: 2D Lösung, allgemein

PDE-System

$$\partial_t \mathbf{u} \;=\; \underset{\sim}{\mathbf{A}}\,\underset{\sim}{\boldsymbol{\nabla}}\mathbf{u}$$

$$\mathbf{u} = \mathbf{u}(t, \underset{\sim}{\mathbf{r}}), \underset{\sim}{\mathbf{A}} = \underset{\sim}{\mathbf{A}}(t, \underset{\sim}{\mathbf{r}}, \mathbf{u})$$

zB. 2D karthesisch

$$\partial_t \mathbf{u} \;=\; \mathbf{A}^x \partial_x \mathbf{u} + \mathbf{A}^y \partial_y \mathbf{u}$$

Lösung PDE-System: setze hier Funktion $p = \underset{\sim}{\mathbf{A}}\underset{\sim}{\boldsymbol{\nabla}}$ als Funktionsoperator

$$\mathbf{u}(t, \underset{\sim}{\mathbf{r}}) \;=\; e^{\int\limits_{t_0}^{t} \underset{\sim}{\mathbf{A}}(\tau)\underset{\sim}{\boldsymbol{\nabla}} d\tau} \mathbf{u}(t_0, \underset{\sim}{\mathbf{r}})$$

$$\begin{aligned} \mathbf{u}(t, \underset{\sim}{\mathbf{r}}) \;&\neq\; \mathbf{u}(t_0, \underset{\sim}{\mathbf{r}} + \underset{\sim}{\mathbf{h}}) \quad !!! \\ &=\; ??? \end{aligned}$$

z.B. System 1D

$$\begin{aligned}\mathbf{u}(t,x) &= e^{\int_{t_0}^{t} \mathbf{A}^x(\tau)\partial_x d\tau} u(t_0,x) \\ &= e^{\mathbf{H}^x \partial_x} \mathbf{u}(t_0,x)\end{aligned}$$

$$\mathbf{H}^x = t\bar{\mathbf{A}}^x = \int_{t_0}^{t} \mathbf{A}^x(\tau)d\tau$$

$$e^{\mathbf{H}^x \partial_x} = \;???$$

Matrixfunktionen

EW-Gleichung

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_m) = 0$$

Cayley-Hamilton

$$(\mathbf{A} - \lambda_1 \mathbf{I})(\mathbf{A} - \lambda_2 \mathbf{I}) \cdots (\mathbf{A} - \lambda_m \mathbf{I}) = 0$$

z.B. m= 3

$$\mathbf{A}^3 = c_2 \mathbf{A}^2 + c_1 \mathbf{A} + c_0 \mathbf{I}$$

Ersatzpolynom (für einfache EW)

$$\begin{aligned}\mathbf{A} &= \mathbf{A}_1\lambda_1 + \mathbf{A}_2\lambda_2 + \cdots + \mathbf{A}_m\lambda_m \\ &= \sum_{i=1}^{m} \mathbf{A}_i\lambda_i \equiv \mathbf{A}_i\lambda_i\end{aligned}$$

mit

$$\mathbf{A}_j\mathbf{A}_k = \mathbf{A}_j\delta_{jk}$$

Matrixfunktionen z.B

$$\mathbf{A}^{-1} = \mathbf{A}_i/\lambda_i$$

$$\mathbf{sin}\mathbf{A} = \mathbf{A}_i\mathbf{sin}\lambda_i$$

$$e^{\mathbf{A}} = \mathbf{A}_i e^{\lambda_i}$$

z.B. für m= 3

$$\mathbf{A}_i = [(\mathbf{A} - \lambda_j \mathbf{I})/(\lambda_i - \lambda_j)][(\mathbf{A} - \lambda_k \mathbf{I})/(\lambda_i - \lambda_k)]$$

z.B. mit Eigenvektoren $\lambda_1 = -a,\ \ \lambda_2 = 0,\ \ \lambda_3 = +a$

$$\mathbf{A}_1 = [(\mathbf{A} - \lambda_2 \mathbf{I})/(\lambda_1 - \lambda_2)][(\mathbf{A} - \lambda_3 \mathbf{I})/(\lambda_1 - \lambda_3)]$$

$$\mathbf{A}_1 = [(\mathbf{A} - 0\mathbf{I})/(-a - 0)][(\mathbf{A} - a\mathbf{I})/(-a - a)]$$

$$\begin{aligned}
\mathbf{A}_1 &= \tfrac{1}{2}(\mathbf{A}/a)^2 - \tfrac{1}{2}(\mathbf{A}/a) \\
\mathbf{A}_2 &= \mathbf{I} - (\mathbf{A}/a)^2 \\
\mathbf{A}_3 &= \tfrac{1}{2}(\mathbf{A}/a)^2 + \tfrac{1}{2}(\mathbf{A}/a)
\end{aligned}$$

also z.B. für m= 3:

$$\begin{aligned}\mathbf{u}(t,x) = e^{t\bar{\mathbf{A}}\partial_x}\mathbf{u}(t_0,x) &= \Big(\bar{\mathbf{A}}_1 e^{t\lambda_1\partial_x} + \bar{\mathbf{A}}_2 e^{t\lambda_2\partial_x} + \bar{\mathbf{A}}_3 e^{t\lambda_3\partial_x}\Big)\mathbf{u}(t_0,x)\\ &= \Big(\mathbf{I} + (\bar{\mathbf{A}}/\bar{a})\tfrac{1}{2}(e^{+t\bar{a}\partial_x} - e^{-t\bar{a}\partial_x}) + \\ &\quad \tfrac{1}{2}(\bar{\mathbf{A}}/\bar{a})^2(e^{+t\bar{a}\partial_x} - 2 + e^{-t\bar{a}\partial_x})\Big)\mathbf{u}(t_0,x)\\ &= \Big(\mathbf{I} + t\bar{\mathbf{A}}\partial_x + \tfrac{1}{2}t^2\bar{\mathbf{A}}^2\partial_x^2 + \mathcal{O}\left(t^3\right)\Big)\mathbf{u}(t_0,x)\end{aligned}$$

$$h^x = t\bar{a}$$

$$\begin{aligned}\mathbf{u}(t,x) &= \mathbf{u}(t_0,x) + (\bar{\mathbf{A}}/\bar{a})\tfrac{1}{2}\Big(\mathbf{u}(t_0,x+h^x) - \mathbf{u}(t_0,x-h^x)\Big) + \\ &\quad \tfrac{1}{2}(\bar{\mathbf{A}}/\bar{a})^2\Big(\mathbf{u}(t_0,x+h^x) - 2\mathbf{u}(t_0,x) + \mathbf{u}(t_0,x-h^x)\Big)\end{aligned}$$

exakt! ⟶vergl. Charakteristikentheorie

Lösung System 1D

$$\begin{aligned}
\mathbf{u}(t,x) &= e^{\int_{t_0}^{t} \mathbf{A}^x(\tau)\partial_x d\tau} \mathbf{u}(t_0,x) \\
&= e^{\mathbf{H}^x \partial_x} \mathbf{u}(t_0,x) \\
&= \mathbf{H}_i^x e^{h_i^x \partial_x} \mathbf{u}(t_0,x)
\end{aligned}$$

$$h_i^x = t\lambda_i^x, \;\; \mathbf{H}_i^x = \bar{\mathbf{A}}_i^x$$

$$\mathbf{u}(t,x) \;=\; \bar{\mathbf{A}}_i^x \mathbf{u}(t_0, x + h_i^x)$$

Lösung System 2D

$$\mathbf{u}(t,x,y) = e^{\int_{t_0}^{t}\left(\mathbf{A}^x(\tau)\partial_x + \mathbf{A}^y(\tau)\partial_y\right)d\tau}\mathbf{u}(t_0,x,y)$$

$$= e^{\left(\mathbf{H}^x\partial_x + \mathbf{H}^y\partial_y\right)}\mathbf{u}(t_0,x,y)$$

<u>nicht</u> kommutativ

$$\neq e^{\mathbf{H}^x\partial_x}\ e^{\mathbf{H}^y\partial_y}\ \mathbf{u}(t_0,x,y) \quad !!!$$

jedoch in 2. Ordnung genau

$$e^{\left(\Delta t\mathbf{A} + \Delta t\mathbf{B}\right)} = \tfrac{1}{2}\left(e^{\Delta t\mathbf{A}}\ e^{\Delta t\mathbf{B}} + e^{\Delta t\mathbf{B}}\ e^{\Delta t\mathbf{A}}\right) + \mathcal{O}\left(t^3\right)$$

Lösung 2.Ordnung System 2D

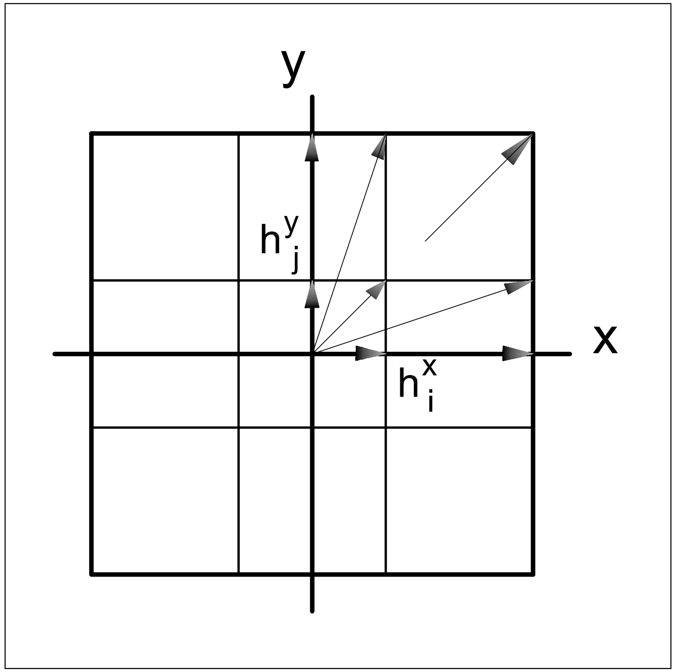
$$\mathbf{u}(t,x,y) \;=\; \tfrac{1}{2}e^{\mathbf{H}^x\partial_x}\; e^{\;\mathbf{H}^y\partial_y}\;\mathbf{u}(t_0,x,y) + \tfrac{1}{2}e^{\mathbf{H}^y\partial_y}\; e^{\;\mathbf{H}^x\partial_x}\;\mathbf{u}(t_0,x,y) + \mathcal{O}\left(t^3\right)$$

Matrixfunktion $\longrightarrow$

$$\bar{\mathbf{A}}_{ij}^{xy} \equiv \tfrac{1}{2}\left(\bar{\mathbf{A}}_i^x\bar{\mathbf{A}}_j^y + \bar{\mathbf{A}}_j^y\bar{\mathbf{A}}_i^x\right)$$

$$\boxed{\mathbf{u}(t,x,y) \;=\; \bar{\mathbf{A}}_{ij}^{xy}\;\mathbf{u}(t_0,x+h_i^x,y+h_j^y) + \mathcal{O}\left(t^3\right)}$$

Lösung 2.Ordnung System 2D



Figur 4: Lösung 2.Ordnung System 2D

Lösung 2.Ordnung System 3D

$$\underline{x} \equiv (x, y, z), \;\; \underline{x}_{ijk} \equiv (x + h_i^x, y + h_j^y, z + h_k^z)$$

$$\bar{\mathbf{A}}_{ijk}^{xyz} \;\equiv\; \tfrac{1}{3}\Big(\bar{\mathbf{A}}_{ij}^{xy}\bar{\mathbf{A}}_{k}^{z} + \bar{\mathbf{A}}_{jk}^{yz}\bar{\mathbf{A}}_{i}^{x} + \bar{\mathbf{A}}_{ki}^{zx}\bar{\mathbf{A}}_{j}^{y}\Big)$$

$$\mathbf{u}(t, \underline{x}) \;=\; \bar{\mathbf{A}}_{ijk}^{xyz}\mathbf{u}(t_0, \underline{x}_{ijk}) + \mathcal{O}\left(t^3\right)$$

Lösung System 3D
hier: koordinatenfrei

$$
\begin{aligned}
\check{\mathbf{u}}(t, \underset{\sim}{\mathbf{r}}) &= e^{\int\limits_{t_0}^{t} \check{\mathbf{A}}(\tau)\underset{\sim}{\boldsymbol{\nabla}} d\tau} \check{\mathbf{u}}(t_0, \underset{\sim}{\mathbf{r}}) \\
&= e^{t\check{\bar{\mathbf{A}}}\underset{\sim}{\boldsymbol{\nabla}}} \check{\mathbf{u}}(t_0, \underset{\sim}{\mathbf{r}}) \\
&= \sum_{i=1}^{m} \check{\bar{\mathbf{A}}}_i e^{\check{\mathbf{h}}_i \underset{\sim}{\boldsymbol{\nabla}}} \check{\mathbf{u}}(t_0, \underset{\sim}{\mathbf{r}})
\end{aligned}
$$

hier $\check{\mathbf{u}}$ und $\check{\mathbf{A}}$ i.a. mit Vektor- und Skalar-Komponenten

Beispiel: lin. Euler-Gleichungen

$$\begin{aligned}
\partial_t \underset{\sim}{\mathbf{v}} + (\underset{\sim}{\bar{\mathbf{v}}}\underset{\sim}{\boldsymbol{\nabla}})\underset{\sim}{\mathbf{v}} &= -\frac{1}{\bar{\rho}}\ \mathbf{grad}(p)\ ;\ \mathbf{curl}(\underset{\sim}{\mathbf{v}}) = \underset{\sim}{\mathbf{0}} \\
\partial_t p + (\underset{\sim}{\bar{\mathbf{v}}}\underset{\sim}{\boldsymbol{\nabla}})p &= -\gamma\bar{p}\ \mathbf{div}(\underset{\sim}{\mathbf{v}}) \\
\partial_t \rho + (\underset{\sim}{\bar{\mathbf{v}}}\underset{\sim}{\boldsymbol{\nabla}})\rho &= -\bar{\rho}\ \mathbf{div}(\underset{\sim}{\mathbf{v}})
\end{aligned}$$

$$\partial_t \check{\mathbf{u}} + (\underset{\sim}{\bar{\mathbf{v}}}\underset{\sim}{\boldsymbol{\nabla}})\check{\mathbf{u}} = \check{\bar{\mathbf{A}}}\underset{\sim}{\boldsymbol{\nabla}}\check{\mathbf{u}}$$

Beispiel: lin. Euler-Gleichungen, m = 3

$$\bar{\mathbf{A}} = - \begin{pmatrix} 0 & 1/\bar{\rho} & 0 \\ \gamma\bar{p} & 0 & 0 \\ \bar{\rho} & 0 & 0 \end{pmatrix}$$

$$\lambda_1 = -\bar{a}, \;\; \lambda_2 = 0, \;\; \lambda_3 = +\bar{a}$$
$$\bar{a}^2 = \gamma\bar{p}/\bar{\rho}$$

$$\bar{\mathbf{A}}^2 = \begin{pmatrix} \bar{a}^2 & 0 & 0 \\ 0 & \bar{a}^2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Beispiel: m = 3

$$
\begin{aligned}
e^{t\bar{\mathbf{A}}\underset{\sim}{\boldsymbol{\nabla}}} &= \bar{\mathbf{A}}_1 e^{t\lambda_1\underset{\sim}{\boldsymbol{\nabla}}} + \bar{\mathbf{A}}_2 e^{t\lambda_2\underset{\sim}{\boldsymbol{\nabla}}} + \bar{\mathbf{A}}_3 e^{t\lambda_3\underset{\sim}{\boldsymbol{\nabla}}} \\
&= \mathbf{I} - (\bar{\mathbf{A}}/\bar{a})^2 + \\
&\quad \tfrac{1}{2}(\bar{\mathbf{A}}/\bar{a})(e^{+\bar{R}\underset{\sim}{\boldsymbol{\nabla}}} - e^{-\bar{R}\underset{\sim}{\boldsymbol{\nabla}}}) + \\
&\quad \tfrac{1}{2}(\bar{\mathbf{A}}/\bar{a})^2(e^{+\bar{R}\underset{\sim}{\boldsymbol{\nabla}}} + e^{-\bar{R}\underset{\sim}{\boldsymbol{\nabla}}})
\end{aligned}
$$

$$\bar{R} := t\bar{a}$$

Beispiel: m = 3

$$\tfrac{1}{2}(e^{+\bar{R}\underset{\sim}{\boldsymbol{\nabla}}} - e^{-\bar{R}\underset{\sim}{\boldsymbol{\nabla}}}) = \bar{R}\underset{\sim}{\boldsymbol{\nabla}} + \tfrac{1}{6}\bar{R}^3\underset{\sim}{\boldsymbol{\nabla}}\boldsymbol{\Delta} + \tfrac{1}{120}\bar{R}^5\underset{\sim}{\boldsymbol{\nabla}}\boldsymbol{\Delta}^2 + \cdots$$

$$= \bar{R}\underset{\sim}{\boldsymbol{\nabla}}(1 + \tfrac{1}{6}\bar{R}^2\boldsymbol{\Delta} + \tfrac{1}{120}\bar{R}^4\boldsymbol{\Delta}^2 + \cdots)$$

Laplace Operator

$$\textbf{div grad}(p) = \underset{\sim}{\boldsymbol{\nabla}}\underset{\sim}{\boldsymbol{\nabla}}p = \boldsymbol{\Delta}p$$

$$\textbf{grad div}(\underset{\sim}{\mathbf{v}}) = \underset{\sim}{\boldsymbol{\nabla}}(\underset{\sim}{\boldsymbol{\nabla}}\underset{\sim}{\mathbf{v}}) = \boldsymbol{\Delta}\underset{\sim}{\mathbf{v}} + \textbf{curl curl}(\underset{\sim}{\mathbf{v}})$$

Beispiel: m = 3
Kugelkoordinaten $(r_3, \vartheta, \varphi)$:

$$\mathbf{\Delta} \;=\; \partial_{r_3}^2 + \tfrac{2}{r_3}\partial_{r_3} + \quad \frac{1}{r_3^2}(\partial_\vartheta^2 + \frac{\cos\vartheta}{\sin\vartheta}\partial_\vartheta + \frac{1}{\sin\vartheta^2}\partial_\varphi^2)$$

$$\lim_{r_3\to 0}\mathbf{\Delta} \;=\; 3\,\partial_{r_3}^2 + \quad \cdots$$

$$\mathbf{\Delta}^2 \;=\; (\partial_{r_3}^2 + \tfrac{4}{r_3}\partial_{r_3})\partial_{r_3}^2 + \quad \cdots$$

$$\lim_{r_3\to 0}\mathbf{\Delta}^2 \;=\; 5\,\partial_{r_3}^4 + \quad \cdots$$

$$\lim_{r_3\to 0}(1 + \tfrac{1}{6}\bar{R}^2\mathbf{\Delta} + \tfrac{1}{120}\bar{R}^4\mathbf{\Delta}^2 + \cdots) \;=\; 1 + \tfrac{1}{2}\bar{R}^2\partial_{r_2}^2 + \tfrac{1}{24}\bar{R}^4\partial_{r_4}^4 + \cdots$$

Beispiel: m = 3
Integration über Kugeloberfläche O_3 mit Radius $\bar{R}$:

$$A_{O_3} = \int_{O_3} do_3 = \int_{\vartheta=0}^{\pi} \int_{\varphi=0}^{2\pi} \bar{R} \sin\vartheta d\vartheta \bar{R} d\varphi$$
$$= 4\pi \bar{R}^2$$

$$\int_{O_3} \frac{1}{2}\left(e^{+\bar{R}\underset{\sim}{\boldsymbol{\nabla}}} - e^{-\bar{R}\underset{\sim}{\boldsymbol{\nabla}}}\right)\check{\mathbf{u}}^{(0)} do_3 = \int_{O_3} \bar{R}\underset{\sim}{\boldsymbol{\nabla}}\check{\mathbf{u}}^{(Q_3)} do_3$$

Beispiel: m = 3

$$\frac{1}{2}(e^{+\bar{R}\underset{\sim}{\boldsymbol{\nabla}}} + e^{-\bar{R}\underset{\sim}{\boldsymbol{\nabla}}}) = 1 + \tfrac{1}{2}\bar{R}^2\boldsymbol{\Delta} + \tfrac{1}{24}\bar{R}^4\boldsymbol{\Delta}^2 + \cdots$$
$$= 1 + \tfrac{1}{2}\bar{R}^2\partial_{r_3}^2 + \tfrac{1}{24}\bar{R}^4\partial_{r_3}^4 + \cdots$$
$$+\tfrac{1}{2}\bar{R}^2 2\partial_{r_3}^2 + \tfrac{1}{24}\bar{R}^4 4\partial_{r_3}^4 + \cdots$$
$$= 1 + \tfrac{1}{2}\bar{R}^2\partial_{r_3}^2 + \tfrac{1}{24}\bar{R}^4\partial_{r_3}^4 + \cdots$$
$$+\bar{R}\partial_{r_3}(\bar{R}\partial_{r_3} + \tfrac{1}{6}\bar{R}^3\partial_{r_3}^3 + \cdots)$$

$$\int_{O_3} \frac{1}{2}(e^{+\bar{R}\underset{\sim}{\boldsymbol{\nabla}}} + e^{-\bar{R}\underset{\sim}{\boldsymbol{\nabla}}})\check{\mathbf{u}}^{(0)}do_3 = \int_{O_3}(1 + \tfrac{1}{2}\bar{R}^2\partial_{r_3}^2 + \tfrac{1}{24}\bar{R}^4{\partial_{r_3}^4}^2 + \cdots)\check{\mathbf{u}}^{(0)}do_3$$
$$+\int_{O_3}\bar{R}\partial_{r_3}(\bar{R}\partial_{r_3} + \tfrac{1}{6}\bar{R}^3\partial_{r_3}^3 + \cdots)\check{\mathbf{u}}^{(0)}do_3$$
$$= \int_{O_3}\Big(\check{\mathbf{u}} + \bar{R}\partial_{r_3}\check{\mathbf{u}}\Big)^{(Q_3)}do_3$$

3D Lösung lin. Euler-Gleichngen:

$$\underset{\sim}{\mathbf{v}}^{(\star)} = \frac{1}{A_{O_3 O_3}} \int \left(\underset{\sim}{\mathbf{v}} + \bar{R}\partial_{r_3}\underset{\sim}{\mathbf{v}} - \frac{\bar{R}}{\bar{\rho}\bar{a}}\mathbf{grad}(p) \right)^{(Q_3)} do_3$$

$$p^{(\star)} = \frac{1}{A_{O_3 O_3}} \int \left(p + \bar{R}\partial_{r_3}p - \bar{R}\bar{\rho}\bar{a}\mathbf{div}(\underset{\sim}{\mathbf{v}}) \right)^{(Q_3)} do_3$$

$$\rho^{(\star)} - \rho^{(P)} + \frac{1}{\bar{a}^2}p^{(P)} = \frac{1}{A_{O_3 O_3}} \int \left(p + \bar{R}\partial_{r_3}p - \bar{R}\bar{\rho}\bar{a}\mathbf{div}(\underset{\sim}{\mathbf{v}}) \right)^{(Q_3)} do_3$$

$$\rho^{(\star)} - \rho^{(P)} = \frac{\bar{\rho}}{\gamma\bar{p}}(p^{(\star)} - p^{(P)})$$

Vergleiche Kirchhoffsche 3D Lösung der Wellengleichung !

Lösung System 2D:

Zylinderkoordinaten (r_2, φ):
3D Lösung mit $\check{\mathbf{u}}(z) = const., \partial_{r_3} = \sin\vartheta\partial_{r_2}$
$r_2 = \bar{R}\sin\vartheta,\;\; dr_2 = \bar{R}\cos\vartheta d\vartheta$

$$A_{O_2} = \int\limits_{A_2} da_2 \;=\; \int\limits_0^{2\pi}\int\limits_0^{\bar{R}} r_2 dr_2 d\varphi$$
$$= \pi\bar{R}^2$$

Lösung System 2D:

$$\begin{aligned}
\frac{1}{4\pi\bar{R}^2}\int\limits_{O_3} \check{\mathbf{u}}^{(Q_3)} do_3 &= \frac{1}{4\pi\bar{R}^2}\int\limits_0^{2\pi}\int\limits_0^{\pi} \check{\mathbf{u}}^{(Q_3)} \bar{R}\sin\vartheta d\vartheta \bar{R} d\varphi \\
&= \frac{1}{4\pi\bar{R}^2} 2\int\limits_0^{2\pi}\int\limits_0^{\bar{R}} \check{\mathbf{u}}^{(Q_2)} \frac{r_2}{\cos\vartheta} dr_2 d\varphi \\
&= \frac{1}{2\pi\bar{R}}\int\limits_{A_2} \frac{\check{\mathbf{u}}^{(Q_2)}}{\sqrt{\bar{R}^2 - r_2^2}} da_2 \\
&= \frac{1}{2\pi\bar{R}^2}\int\limits_{A_2} \frac{\check{\mathbf{u}}^{(Q_2)}}{\sqrt{1-(r_2/\bar{R})^2}} da_2 \\
&= \frac{1}{2A_{O_2}}\int\limits_{A_2} \frac{\check{\mathbf{u}}^{(Q_2)}}{\sqrt{1-(r_2/\bar{R})^2}} da_2
\end{aligned}$$

Lösung System 2D:

$$
\begin{aligned}
\frac{1}{4\pi\bar{R}^2}\bar{R}\int\limits_{O_3}\partial_{r_3}\check{\mathbf{u}}^{(Q_3)}do_3 &= \frac{1}{4\pi\bar{R}^2}\bar{R}\int\limits_0^{2\pi}\int\limits_0^{\pi}\partial_{r_3}\check{\mathbf{u}}^{(Q_3)}\bar{R}\sin\vartheta d\vartheta\bar{R}d\varphi \\
&= \frac{1}{4\pi\bar{R}^2}2\bar{R}\int\limits_0^{2\pi}\int\limits_0^{\bar{R}}\partial_{r_2}\check{\mathbf{u}}^{(Q_2)}\frac{r_2\sin\vartheta}{\cos\vartheta}dr_2d\varphi \\
&= \frac{1}{2\pi\bar{R}}\int\limits_{A_2}\frac{r_2\partial_{r_2}\check{\mathbf{u}}^{(Q_2)}}{\sqrt{\bar{R}^2-r_2^2}}da_2 \\
&= \frac{1}{2\pi\bar{R}^2}\int\limits_{A_2}\frac{r_2\partial_{r_2}\check{\mathbf{u}}^{(Q_2)}}{\sqrt{1-(r_2/\bar{R})^2}}da_2 \\
&= \frac{1}{2A_{O_2}}\int\limits_{A_2}\frac{r_2\partial_{r_2}\check{\mathbf{u}}^{(Q_2)}}{\sqrt{1-(r_2/\bar{R})^2}}da_2
\end{aligned}
$$

2D Lösung lin. Euler-Gleichngen:

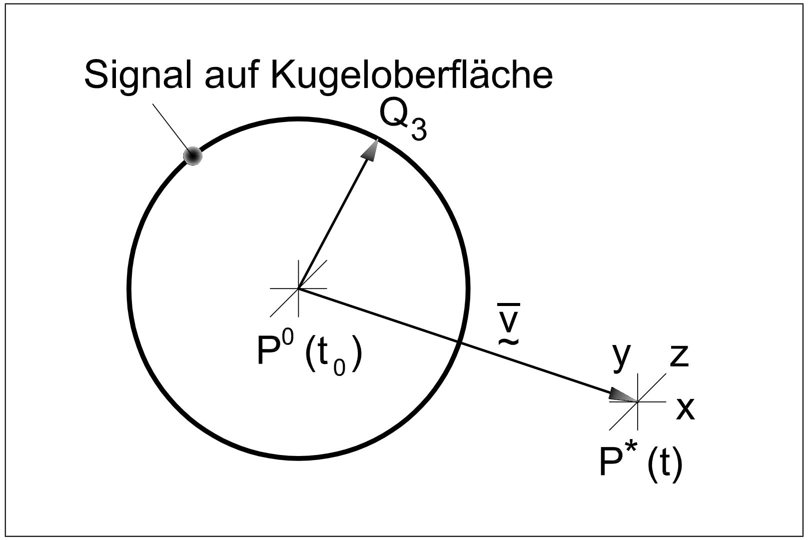
$$\underset{\sim}{\mathbf{v}}^{(\star)} = \frac{1}{2A_{O_2A_2}} \int \frac{\left(\underset{\sim}{\mathbf{v}} + r_2 \partial_{r_2} \underset{\sim}{\mathbf{v}} - \frac{r_2}{\bar{\rho}\bar{a}} \mathbf{grad}(p) \right)^{(Q_2)}}{\sqrt{1 - (r_2/\bar{R})^2}} da_2$$

$$p^{(\star)} = \frac{1}{2A_{O_2A_2}} \int \frac{\left(p + r_2 \partial_{r_2} p - r_2 \bar{\rho}\bar{a} \mathbf{div}(\underset{\sim}{\mathbf{v}}) \right)^{(Q_2)}}{\sqrt{1 - (r_2/\bar{R})^2}} da_2$$

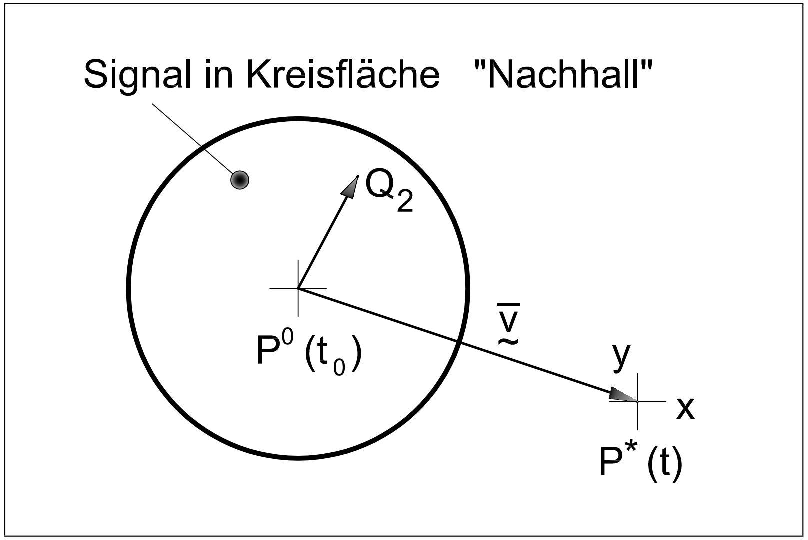
$$\rho^{(\star)} - \rho^{(P)} = \frac{\bar{\rho}}{\gamma \bar{p}} (p^{(\star)} - p^{(P)})$$

Vergleiche Poissonsche 2D Lösung der Wellengleichung !

Lösung lin. Euler-Gleichngen:



Figur 5: 3D Lösung



Figur 6: 2D Lösung

Lösung System 1D:

Koordinate (x):

2D Lösung mit $\check{\mathbf{u}}(y) = const., \partial_{r_2} = \cos\varphi\partial_x$

$$x = r_2 \cos\varphi$$

$$dx = r_2 \sin\varphi d\varphi$$

Lösung System 1D:

$$\begin{aligned}
\frac{1}{2\pi\bar{R}}\int_{A_2} \frac{\check{\mathbf{u}}^{(Q_2)}}{\sqrt{\bar{R}^2 - r_2^2}} da_2 &= \frac{1}{2\pi\bar{R}}\int_0^{\bar{R}}\int_0^{2\pi} \frac{\check{\mathbf{u}}^{(Q_2)}}{\sqrt{\bar{R}^2 - r_2^2}} r_2 d\varphi dr_2 \\
&= \frac{1}{2\pi\bar{R}}\int_{-\bar{R}}^{\bar{R}}\int_{-\sqrt{\bar{R}^2-x^2}}^{+\sqrt{\bar{R}^2-x^2}} \frac{\check{\mathbf{u}}^{(Q_2)}}{\sqrt{\bar{R}^2 - x^2 - y^2}} dx dy \\
&= \frac{1}{2\pi\bar{R}}\int_{-\bar{R}}^{\bar{R}} \left[\arcsin \frac{y}{\sqrt{\bar{R}^2-x^2}}\right]_{-\sqrt{\bar{R}^2-x^2}}^{+\sqrt{\bar{R}^2-x^2}} \check{\mathbf{u}}^{(Q_1)} dx \\
&= \frac{1}{2\bar{R}}\int_{-\bar{R}}^{\bar{R}} \check{\mathbf{u}}^{(Q_1)} dx
\end{aligned}$$

Lösung System 1D:

$$\frac{1}{2\pi\bar{R}}\int_{A_2}\frac{r_2\partial_{r_2}\check{\mathbf{u}}^{(Q_2)}}{\sqrt{\bar{R}^2-r_2^2}}da_2 = \frac{1}{2\bar{R}}\int_{-\bar{R}}^{\bar{R}} x\partial_x\check{\mathbf{u}}^{(Q_1)}dx$$

$$= \frac{1}{2}(u^{(\bar{R})}+u^{(-\bar{R})})-\frac{1}{2\bar{R}}\int_{-\bar{R}}^{\bar{R}}\check{\mathbf{u}}^{(Q_1)}dx$$

1D Lösung lin. Euler-Gleichngen:

$$
\begin{aligned}
{v_x}^{(\star)} &= \frac{1}{2}({v_x}^{(\bar{R})} + {v_x}^{(-\bar{R})}) - \frac{1}{2\bar{\rho}\bar{a}}(p^{(\bar{R})} - p^{(-\bar{R})}) \\
p^{(\star)} &= \frac{1}{2}(p^{(\bar{R})} + p^{(-\bar{R})}) - \frac{\bar{\rho}\bar{a}}{2}({v_x}^{(\bar{R})} - {v_x}^{(-\bar{R})})
\end{aligned}
$$

$$
\rho^{(\star)} - \rho^{(P)} = \frac{\bar{\rho}}{\gamma\bar{p}}(p^{(\star)} - p^{(P)})
$$

Vergleiche d'Alembertsche 1D Lösung der Wellengleichung !

Lösung 3D kugelsymmetrisch, $r_3 \neq 0$: für $v_r(t_0, r_3) \equiv 0$

$$
\begin{aligned}
p(t, r_3) &= \tfrac{1}{2}\left(e^{+\bar{R}\underset{\sim}{\boldsymbol{\nabla}}} + e^{-\bar{R}\underset{\sim}{\boldsymbol{\nabla}}}\right)p(t_0, r_3) \\
&= (1 + \tfrac{1}{2}\bar{R}^2\boldsymbol{\Delta} + \tfrac{1}{24}\bar{R}^4\boldsymbol{\Delta}^2 + \cdots)p(t_0, r_3) \\
&= \Big(1 + \tfrac{1}{2}\bar{R}^2(\partial_{r_3}^2 + \tfrac{2}{r_3}\partial_{r_3}) + \tfrac{1}{24}\bar{R}^4(\partial_{r_3}^4 + \tfrac{4}{r_3}\partial_{r_3}^3) + \cdots\Big)p(t_0, r_3) \\
&= \tfrac{1}{2}(p(t_0, r_3 + \bar{R}) + p(t_0, r_3 - \bar{R})) + \\
&\quad \tfrac{\bar{R}}{2r_3}(p(t_0, r_3 + \bar{R}) - p(t_0, r_3 - \bar{R})) \\
&= \tfrac{r_3 + \bar{R}}{2r_3}p(t_0, r_3 + \bar{R}) + \tfrac{r_3 - \bar{R}}{2r_3}p(t_0, r_3 - \bar{R})
\end{aligned}
$$

Lösung 3D kugelsymmetrisch, $r_3 \neq 0$: für $p(t_0, r_3) \equiv 0$

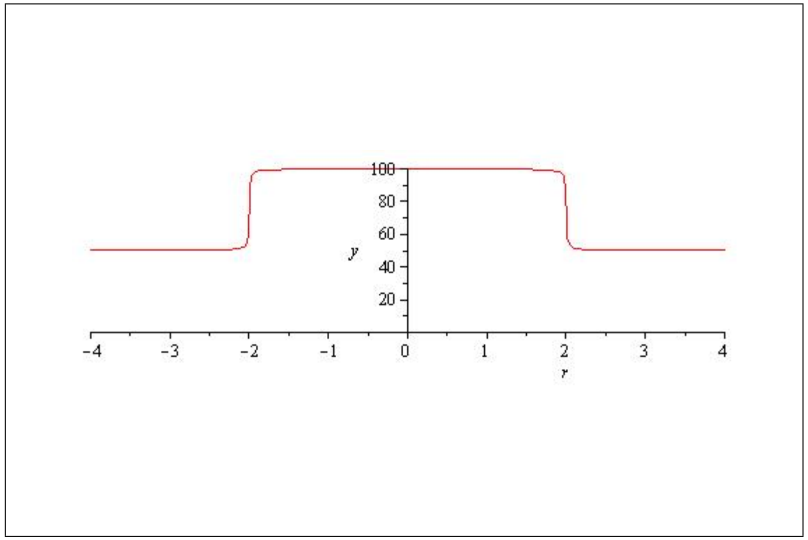
$$
\begin{aligned}
p(t, r_3) &= -\frac{\bar{\rho}\bar{a}}{2}\left(e^{+\bar{R}\underset{\sim}{\boldsymbol{\nabla}}} - e^{-\bar{R}\underset{\sim}{\boldsymbol{\nabla}}}\right)\underset{\sim}{\mathbf{v}}(t_0, r_3) \\
&= -\bar{\rho}\bar{a}\bar{R}\left(1 + \tfrac{1}{6}\bar{R}^2\boldsymbol{\Delta} + \tfrac{1}{120}\bar{R}^4\boldsymbol{\Delta}^2 + \cdots\right)\left(\partial_{r_3} + \tfrac{2}{r_3}\right)v_r(t_0, r_3) \\
&= -\bar{\rho}\bar{a}\bar{R}\left(\partial_{r_3} + \tfrac{2}{r_3}\right)v_r(t_0, r_3) - \frac{\bar{\rho}\bar{a}}{6}\bar{R}^3\left(\partial_{r_3} + \tfrac{4}{r_3}\right)\partial_{r_3}^2 v_r(t_0, r_3) \\
&\quad -\frac{\bar{\rho}\bar{a}}{120}\bar{R}^5\left(\partial_{r_3} + \tfrac{6}{r_3}\right)\partial_{r_3}^4 v_r(t_0, r_3) + \cdots \\
&= -\bar{\rho}\bar{a}\left(\bar{R}\partial_{r_3} + \tfrac{1}{6}\bar{R}^3\partial_{r_3}^3 + \tfrac{1}{120}\bar{R}^5\partial_{r_3}^5 + \cdots\right)v_r(t_0, r_3) \\
&\quad -\bar{\rho}\bar{a}\frac{\bar{R}}{r_3}\left(1 + \tfrac{1}{2}\bar{R}^2\partial_{r_3}^2 + \tfrac{1}{24}\bar{R}^4\partial_{r_3}^4 + \cdots\right)v_r(t_0, r_3) \\
&\quad -\bar{\rho}\bar{a}\frac{1}{r_3}\left(\bar{R} + \tfrac{1}{6}\bar{R}^3\partial_{r_3}^2 + \tfrac{1}{120}\bar{R}^5\partial_{r_3}^4 + \cdots\right)v_r(t_0, r_3)
\end{aligned}
$$

$$
\begin{aligned}
p(t, r_3) \;=\;& -\bar{\rho}\bar{a}(\bar{R}\partial_{r_3} + \tfrac{1}{6}\bar{R}^3\partial_{r_3}^3 + \tfrac{1}{120}\bar{R}^5\partial_{r_3}^5 + \cdots)v_r(t_0, r_3) \\
& -\bar{\rho}\bar{a}\frac{\bar{R}}{r_3}(1 + \tfrac{1}{2}\bar{R}^2\partial_{r_3}^2 + \tfrac{1}{24}\bar{R}^4\partial_{r_3}^4 + \cdots)v_r(t_0, r_3) \\
& -\bar{\rho}\bar{a}\frac{1}{r_3}\int_0^{\bar{R}}(1 + \tfrac{1}{2}r^2\partial_{r_3}^2 + \tfrac{1}{24}r^4\partial_{r_3}^4 + \cdots)v_r(t_0, r_3)dr \\
=\;& -\bar{\rho}\bar{a}\left(\tfrac{r_3+\bar{R}}{2r_3}v_r(t_0, r_3 + \bar{R}) - \tfrac{r_3-\bar{R}}{2r_3}v_r(t_0, r_3 - \bar{R})\right) \\
& -\bar{\rho}\bar{a}\frac{1}{2r_3}\int_{-\bar{R}}^{\bar{R}} v_r(t_0, r_3 + r)dr
\end{aligned}
$$

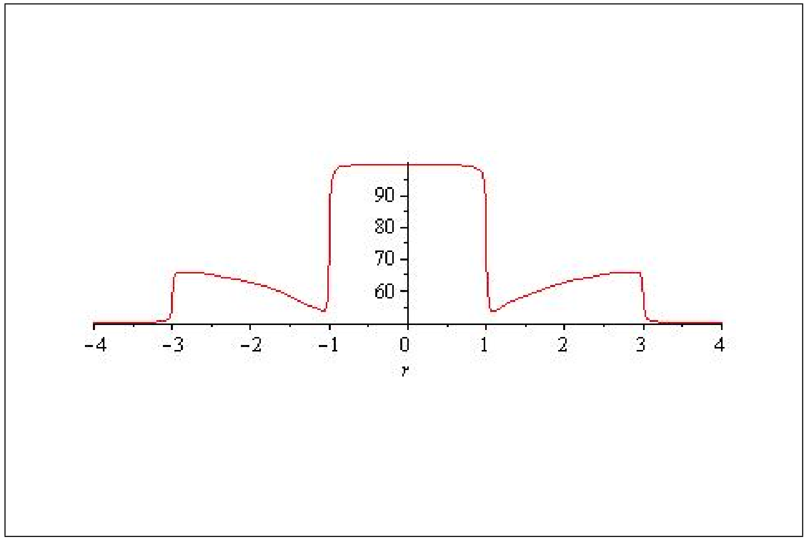
3D Lösung lin. Euler-Gleichngen, kugelsymmetrisch:

$$
\begin{aligned}
p(t,r_3) \;=\; & \tfrac{r_3+\bar{R}}{2r_3} p(t_0, r_3+\bar{R}) + \tfrac{r_3-\bar{R}}{2r_3} p(t_0, r_3-\bar{R}) \\
& -\bar{\rho}\bar{a}\left(\tfrac{r_3+\bar{R}}{2r_3} v_r(t_0, r_3+\bar{R}) - \tfrac{r_3-\bar{R}}{2r_3} v_r(t_0, r_3-\bar{R}) \right) \\
& -\bar{\rho}\bar{a}\frac{1}{2r_3} \int\limits_{-\bar{R}}^{\bar{R}} v_r(t_0, r_3+r)\,dr
\end{aligned}
$$

Lösung lin. Euler-Gleichngen, kugelsymmetrisch AW p(t=0), v(t=0)=0

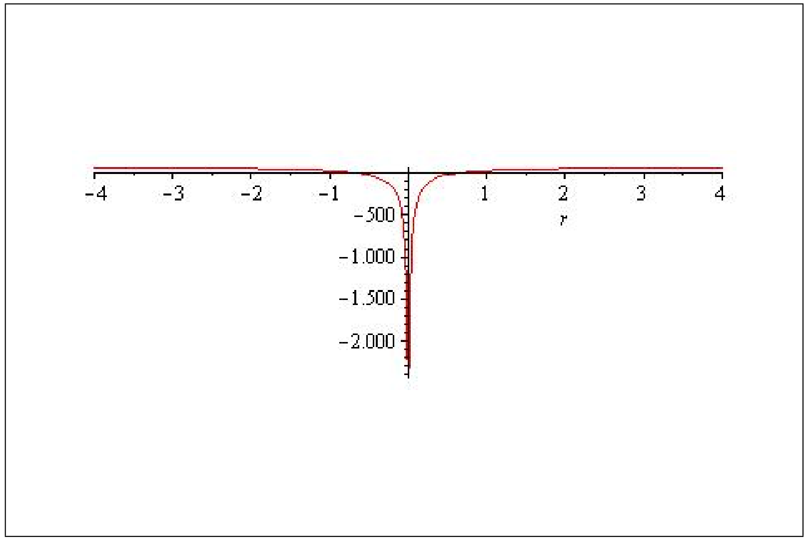


Figur 7: Anfangswerte p(t=0)

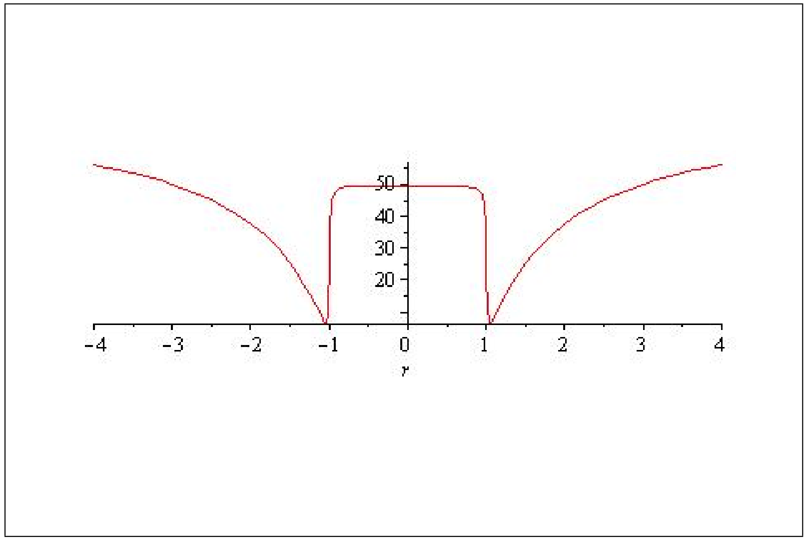


Figur 8: Lösung p(t=1), AW p(t=0), v(t=0)=0

Lösung lin. Euler-Gleichngen, kugelsymmetrisch AW p(t=0), v(t=0)=0

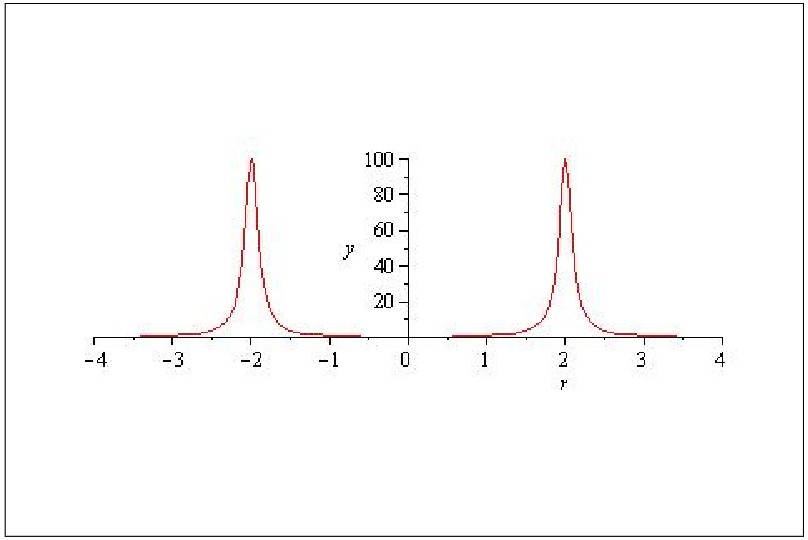


Figur 9: p(t=2), AW p(t=0), v(t=0)=0

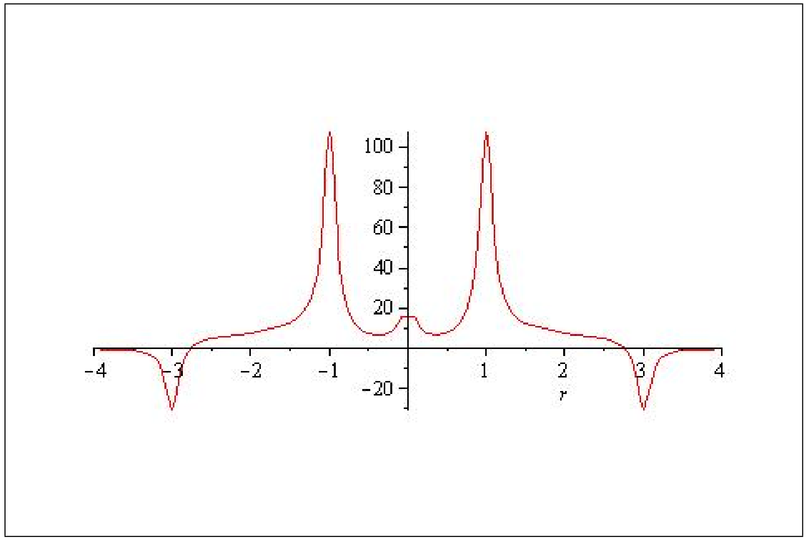


Figur 10: Lösung p(t=3), AW p(t=0), v(t=0)=0

Lösung lin. Euler-Gleichngen, kugelsymmetrisch AW v(t=0), p(t=0)=0

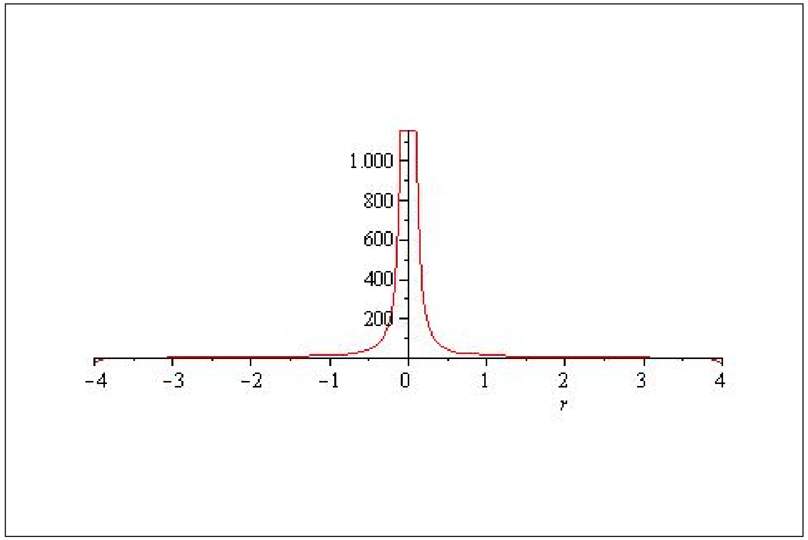


Figur 11: Anfangswerte v(t=0)

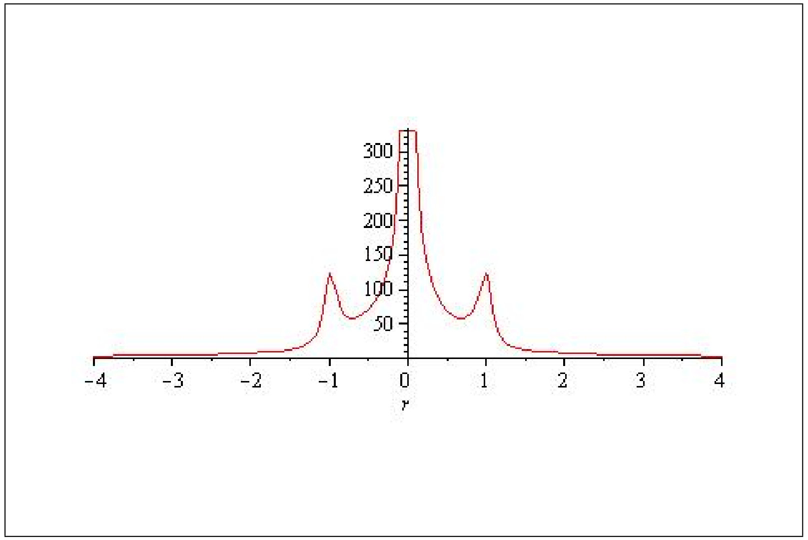


Figur 12: Lösung p(t=1), AW v(t=0), p(t=0)=0

Lösung lin. Euler-Gleichngen, kugelsymmetrisch AW v(t=0), p(t=0)=0



Figur 13: p(t=2), AW v(t=0), p(t=0)=0



Figur 14: Lösung p(t=3), AW v(t=0), p(t=0)=0